# 3 Sobolev Spaces

**Exercise 3.1.** Let $u \in \mathrm{Lip}(\Omega)$ and denotes by $L$ its best Lipschitz constant. Then we can extend it to the whole $\mathbb{R}^d$ by letting

$$\widetilde{u}(x) = \inf\{u(y) + L|x-y| : y \in \Omega\}.$$

We want to prove that

$$v \in \mathrm{Lip}(\mathbb{R}^d) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^d) \Rightarrow Dv \in L^\infty(\mathbb{R}^d). \tag{1}$$

Once this is done, then we can conclude :

$$u \in \mathrm{Lip}(\Omega) \cap L^\infty(\Omega) \Rightarrow \widetilde{u} \in \mathrm{Lip}(\mathbb{R}^d) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^d) \Rightarrow D\widetilde{u} \in L^\infty(\mathbb{R}^) \Rightarrow Du \in L^\infty(\Omega)$$

and therefore $u \in W^{1,\infty}(\Omega)$.

We now prove (1). Let $\rho_\varepsilon$ be standard mollifiers and $v_\varepsilon = v * \rho_\varepsilon$. Then $v_\varepsilon$ is $L$-Lipschitz since

$$|v_\varepsilon(x) - v_\varepsilon(y)| \leq \int_{\mathbb{R}^d} |v(x-t) - v(y-t)|\rho_\varepsilon(t)dt \leq L|x-y| \int_{\mathbb{R}^d} \rho_\varepsilon = L|x-y|.$$

Since $v_\varepsilon$ is smooth, we know by the classical theory that $\|Dv_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq L$. In particular $\|\partial_i v_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq L$ for every $i = 1, \dots, n$ and every $\varepsilon > 0$. By Banach-Alaoglu theorem there exists a sequence $\varepsilon_k \to 0$ and $z_i \in L^\infty(\mathbb{R}^d)$ such that $\partial_i v_{\varepsilon_k} \to z_i$ weakly* in $L^\infty(\mathbb{R}^d)$. We prove that $z_i = \partial_i v$, being $z_i \in L^\infty(\mathbb{R}^d)$, this will conclude the proof. For every $\varphi \in C^\infty_c(\mathbb{R}^d)$ and every $k \in \mathbb{N}$ it holds

$$\int_{\mathbb{R}^d} v_{\varepsilon_k} \partial_i \varphi dx = -\int_{\mathbb{R}^d} \varphi \partial_i v_{\varepsilon_k} dx.$$

Letting $k \to \infty$ we get

$$\int_{\mathbb{R}^d} v \partial_i \varphi dx = -\int_{\mathbb{R}^d} \varphi z_i dx,$$

since $v_{\varepsilon_k} \to v$ uniformly and therefore in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and since $\partial_i v_{\varepsilon_k} \to z_i$ weakly* in $L^\infty(\mathbb{R}^d)$. This shows that $z_i = \partial_i v$ and concludes the proof.

**Exercise 3.2.** Since $u \in W^{1,p}_0(\Omega)$, there exists a sequence $u_\varepsilon \in C^\infty_c(\Omega)$ such that $u_\varepsilon \to u$ in $W^{1,p}(\Omega)$. From the proof of the chain rule formula we deduce that $G(u_\varepsilon) \to G(u)$ in $W^{1,p}(\Omega)$, thus it is enough to prove that $G(u_\varepsilon) \in W^{1,p}_0(\Omega)$ for every $\varepsilon > 0$. Since $u_\varepsilon \in C^\infty_c(\Omega)$ and $G(0) = 0$ we get that $\mathrm{supp}G(u_\varepsilon)$ is compactly contained in $\Omega$, which together with $G(u_\varepsilon) \in W^{1,p}(\Omega)$, give $G(u_\varepsilon) \in W^{1,p}_0(\Omega)$. Indeed we could just mollify the whole function $G(u_\varepsilon)$, getting a sequence $(G(u_\varepsilon))_\delta \in C^\infty_c(\Omega)$ such that $(G(u_\varepsilon))_\delta \to G(u_\varepsilon)$ in $W^{1,p}(\Omega)$, as $\delta \to 0$.

**Exercise 3.3.** Let $u(x) = |x|^{-\alpha}$, $\alpha > 0$. Note that $u \notin C^0(\Omega)$ since $u(x) \to +\infty$ as $x \to 0$. We have

$$\int_{B_1} |u(x)|^2\,dx = \int_{B_1} \frac{1}{|x|^{2\alpha}}\,dx = 2\pi \int_0^1 \frac{1}{r^{2\alpha-1}}\,dr.$$

Thus $u \in L^2(\Omega)$ if and only if $\alpha < 1$. We now compute its weak derivative. Let $\varphi \in C_c^\infty(\Omega;\mathbb{R}^2)$, we have

$$\begin{aligned}\langle \nabla u, \varphi\rangle &= -\langle u, \operatorname{div}\varphi\rangle = -\lim_{\varepsilon\to 0}\int_{B_1\setminus B_\varepsilon} u(x)\operatorname{div}\varphi(x)\,dx \\ &= -\lim_{\varepsilon\to 0}\left(\int_{\partial B_\varepsilon} u(x)\varphi(x)\cdot n\,d\sigma - \int_{B_1\setminus B_\varepsilon} \nabla u(x)\cdot\varphi(x)\,dx\right). \end{aligned} \tag{2}$$

The first term can be estimated as

$$\left|\int_{\partial B_\varepsilon} u(x)\varphi(x)\cdot n\,d\sigma\right| C\varepsilon^{1-\alpha}\|\varphi\|_{L^\infty(B_1)}, \tag{3}$$

that goes to zero, as $\varepsilon \to 0$, since $\alpha < 1$. Moreover $u \in C^1(B_1 \setminus B_\varepsilon)$, thus

$$\nabla u(x) = -\alpha\frac{x}{|x|^{\alpha+2}}, \ \text{ in } B_1 \setminus B_\varepsilon.$$

Thus, in order to have $\nabla u$ at least in $L^1$, we need to have $x|x|^{-2-\alpha} \in L^1(B_1)$, which amounts to

$$\int_{B_1} \frac{1}{|x|^{\alpha+1}}\,dx = 2\pi\int_0^1 \frac{1}{r^\alpha} < \infty,$$

from which, together with (2) and (3), we also deduce that $u \in W^{1,1}(\Omega)$ with $\nabla u = -\alpha\frac{x}{|x|^{\alpha+2}}$. Finally,

$$\int_{B_1} |\nabla u(x)|^2\,dx = \alpha^2\int_{B_1}\frac{1}{|x|^{2\alpha+2}}\,dx = 2\pi\alpha^2\int_0^1 \frac{1}{r^{2\alpha+1}}\,dr,$$

and this is finite if and only if $\alpha < 0$, from which we conclude that it does not exists $\alpha > 0$ such that $u \in H^1(\Omega)$.

Note that by the previous computations we also get $u \in W^{1,p}(\Omega)$ for all $p < 2$ if $\alpha < \frac{2-p}{p} < 1$.

**Exercise 3.4.** Note that the functions $f(x) = \max\{x, 0\}$ and $g(x) = \max\{-x, 0\}$ are Lipschitz functions from $\mathbb{R}$ to $\mathbb{R}$. Moreover

$$f'(x) = \chi_{x>0} \quad \text{and} \quad g'(x) = -\chi_{x<0}.$$

So that $u^+ = f(u), u^- = g(u) \in W^{1,p}(\Omega)$ and moreover $\partial_i u^+ = f'(u)\partial_i u = \chi_{u>0}\partial_i u$ and $\partial_i u^- = -\chi_{u<0}\partial_i u$, from which we also deduce $|u| = u^+ + u^- \in W^{1,p}(\Omega)$ with

$$\partial_i |u| = (\chi_{u>0} - \chi_{u<0})\partial_i u.$$

We now note that $\min\{u, M\} = \min\{u - M, 0\} + M = -\max\{M - u, 0\} + M$ and $\max\{u, M\} = \max\{u - M, 0\} + M$. Thus by the previous computations we deduce that the truncated function $T_M u \in W^{1,p}(\Omega)$.

**Exercise 3.5.** We compute

$$\begin{aligned}\partial_i^h u(x) - \partial_i u(x) &= \frac{u(x + he_i) - u(x)}{h} - \partial_i u(x) = \frac{1}{h}\int_0^1 \frac{d}{dt} u(x + the_i)\, dt - \partial_i u(x) \\ &= \int_0^1 \nabla u(x + the_i) \cdot e_i \, dt - \partial_i u(x) = \int_0^1 (\partial_i u(x + the_i) - \partial_i u(x))\, dt,\end{aligned}$$

and by Jensen's inequality

$$\|\partial_i^h u - \partial_i u\|_{L^p(\mathbb{R}^d)}^p \leq \int_0^1 \int_{\mathbb{R}^d} |\partial_i u(x + the_i) - \partial_i u(x))|^p \, dxdt\,. \tag{4}$$

Note that the computations we did to show the previous inequality required that $u$ was smooth (or at least $C^1$). Thus, to be precise, one at first proves that (4) holds for every smooth function $u$, and then by density, it holds for every $u \in W^{1,p}(\mathbb{R}^d)$.

Since the translation operator is continuous in $L^p(\mathbb{R}^d)$ we get that

$$\lim_{h\to 0} \int_{\mathbb{R}^d} |\partial_i u(x + the_i) - \partial_i u(x))|^p \, dx \to 0$$

for almost every $t \in (0, 1)$, and moreover

$$\int_{\mathbb{R}^d} |\partial_i u(x + the_i) - \partial_i u(x))|^p \, dx \leq C \int_{\mathbb{R}^d} |\partial_i u(x)|^p \, dx \in L^1(0, 1).$$

Thus, by the Lebesgue dominated convergence, we conclude

$$\lim_{h\to 0} \|\partial_i^h u - \partial_i u\|_{L^p(\mathbb{R}^d)} = 0.$$